Decomposing simple permutations, with enumerative consequences

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Abstract

We prove that every sufficiently long simple permutation contains two long almost disjoint simple subsequences. This result has applications to the enumeration of restricted permutations. For example, it immediately implies a result of Bóna and (independently) Mansour and Vainshtein that for any r, the number of permutations with at most r copies of 132 has an algebraic generating function.

1 Statement of theorem

Simplicity, under a variety of names[1], has been studied for a wide range of combinatorial objects. Our main result concerns simple permutations; possible analogues for other contexts are discussed in the conclusion. An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ also forms an interval of natural numbers. Every permutation π of $[n] = \{1, 2, \ldots, n\}$ has intervals of size 0, 1, and n; π is said to be *simple* if it has no other intervals. Figure 1 shows the plots of two simple permutations. Intervals of permutations are interesting in their own right and have applications to biomathematics; see Corteel, Louchard, and Pemantle [10], where among other results it is proved that the number of simple permutations of $[n]$ is asymptotic to $n!/e^2$. More precise asymptotics are given by Albert, Atkinson, and Klazar [2].

Each sequence of distinct real numbers is *order isomorphic* to a unique permutation; this is the permutation with the same relative comparisons. We say that a sequence of distinct

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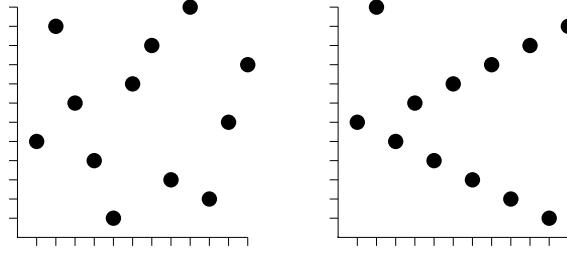[1]Two synonyms for simplicity are primality and indecomposability.

Figure 1: The plots of two simple permutations. Note that every simple subsequence of the permutation on the right must contain its first two entries.

real numbers is simple if it is order isomorphic to a simple permutation. We prove that long simple permutations must contain two long almost disjoint simple subsequences. Formally:

Theorem 1.1. *There is a function $f(k)$ such that every simple permutation of length at least $f(k)$ contains two simple subsequences, each of length at least k, sharing at most two entries.*

The second "two" in the statement of Theorem 1.1 is best possible, as is demonstrated by the family of simple permutations of the form $m(2m)(m-1)(m+1)(m-2)(m+2)\cdots 1(2m-1)$; the permutation on the right of Figure 1 is of this form. On the other hand, no attempt has been made to optimise the function f; our proof gives an f of order about k^{k^k}. The implications of Theorem 1.1 are discussed in the next section. The proof begins in Section 3.

2 Implications/motivation

The permutation π is said to *contain* the permutation σ, written $\sigma \leq \pi$, if π has a subsequence that is order isomorphic to σ. For example, $\pi = 391867452$ contains $\sigma = 51342$, as can be seen by considering the subsequence $91672 (= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$, and such a subsequence is called a *copy* of σ in π. This pattern-containment relation is a partial order on permutations. We refer to downsets of permutations under this order as *permutation classes*. In other words, if $\mathcal{C}$ is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. We denote by $\mathcal{C}_n$ the set $\mathcal{C} \cap S_n$, i.e. the permutations in $\mathcal{C}$ of length n, and we refer to $\sum |\mathcal{C}_n| x^n$ as the *generating function for $\mathcal{C}$*. Recall that an *antichain* is a set of pairwise incomparable elements. For any permutation class $\mathcal{C}$, there is a unique (possibly infinite) antichain B such that $\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$. This antichain B, which consists of the minimal permutations not in $\mathcal{C}$, is called the *basis* of $\mathcal{C}$.

In a class with only finitely many simple permutations, long permutations must map nontrivial intervals onto intervals. Thus these classes have a recursive structure in which long permutations are built up from smaller permutations, and so it is natural to expect them to have algebraic generating functions. This is indeed the case:

Theorem 2.1 (Albert and Atkinson [1]). *A permutation class with only finitely many simple permutations has a readily computable algebraic generating function.*

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One of the simplest classes with only finitely many simple permutations is $\mathrm{Av}(132)^2$. Theorems 1.1 and 2.1 combine to give a short proof of the following result.

Theorem 2.2 (Bóna [5]; Mansour and Vainshtein [17]). *For every r, the class of all permutations containing at most r copies of 132 has an algebraic generating function*[3].

Proof of Theorem 2.2 via Theorems 1.1 and 2.1. We wish to show that only finitely many simple permutations contain at most r copies of 132, or in other words, that there is a function $g(r)$ so that every simple permutation of length at least $g(r)$ contains more than r copies of 132. Footnote 2 shows that we may take $g(0) = 3$. We now proceed by induction, setting $g(r) = f(g(\lfloor r/2 \rfloor))$, where f is the function from Theorem 1.1. By that theorem, every simple permutation π of length at least $g(r)$ contains two simple subsequences of length at least $g(\lfloor r/2 \rfloor)$. By induction each of these simple subsequences contains more than $\lfloor r/2 \rfloor$ copies of 132. Moreover, because these simple subsequences share at most two entries, their copies of 132 are distinct, and thus π contains more than r copies of 132, as desired. □

Indeed, the proof above shows that every permutation class whose members contain a bounded number of copies of 132 has an algebraic generating function, whereas Theorem 2.2 is concerned only with the entire class of permutations with at most r copies of 132. There is of course nothing special about 132. Denote by $\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$ the class of permutations that have at most r_1 copies of β_1, at most r_2 copies of β_2, and so on[4]. The proof just given can be adapted to prove the following result.

Corollary 2.3. *If the class $\mathrm{Av}(\beta_1, \beta_2, \ldots, \beta_k)$ contains only finitely many simple permutations then for all choices of nonnegative integers r_1, r_2, ..., and r_k, the class $\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$ also contains only finitely many simple permutations.*

The largest permutation class whose only simple permutations are 1, 12, and 21 is the class of *separable permutations*, $\mathrm{Av}(2413, 3142)$. Thus as another instance of Corollary 2.3, we have the following.

Corollary 2.4. *For all r and s, every subclass of $\mathrm{Av}(2413^{\leq r}, 3142^{\leq s})$ contains only finitely many simple permutations and thus has an algebraic generating function.*

Theorem 1.1 does not apply only to permutation classes. In Brignall, Huczynska, and Vatter [7], Theorem 2.1 is extended to "finite query-complete sets of properties". As a specialisation of that theorem, we have the following.

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[2]In any permutation from $\mathrm{Av}(132)$, all entries to the left of the maximum must be greater than all entries to the right. This shows that $\mathrm{Av}(132)$ has only three simple permutations (1, 12, and 21).

[3]For example, the generating function in the $r = 1$ case is

$$\frac{1 - \sqrt{1 - 4x}}{2x} + \frac{8x^3}{\sqrt{1 - 4x}\left(1 + \sqrt{1 - 4x}\right)^3}$$

(due, originally, to Bóna [6]).

[4]That this is a permutation class is clear, although finding its basis may be less obvious. An easy argument shows that the basis elements of this class have length at most $\max\{(r_i + 1)|\beta_i| : i \in [k]\}$; see Atkinson [3] for the details. One such computation: $\mathrm{Av}(132^{\leq 1}) = \mathrm{Av}(1243, 1342, 1423, 1432, 2143, 35142, 354162, 461325, 465132)$.

Theorem 2.5 (Brignall, Huczynska, and Vatter [7]). *In a permutation class $\mathcal{C}$ with only finitely many simple permutations, the following sequences have algebraic generating functions:*

- *the number of even permutations in $\mathcal{C}_n$,*

- *the number of involutions in $\mathcal{C}_n$,*

- *the number of even involutions in $\mathcal{C}_n$,*

- *the number of alternating permutations in $\mathcal{C}_n$,*

- *the number of permutations in $\mathcal{C}_n$ avoiding a finite set of blocked permutations[5].*

There are several results in the literature that follow from the combination of Theorems 1.1, and 2.5:

- Even permutations in $\mathrm{Av}(132^{\leq r})$ — Mansour [16].

(When counting even permutations, unlike when counting all permutations, symmetry considerations reduce us to three cases of length three permutations — 123, 132, and 231 — not two[6], and thus there is another result we can state: the even permutations in $\mathrm{Av}(231^{\leq r})$ have an algebraic generating function for all r, although this result seems to have escaped print.)

- Involutions in $\mathrm{Av}(231^{\leq r})$ — Mansour, Yan, and Yang [18]. In the same reference: even involutions in $\mathrm{Av}(231^{\leq r})$. The case of involutions in $\mathrm{Av}(132^{\leq r})$ is due to Mansour [private communication].

- Alternating permutations in $\mathrm{Av}(132^{\leq r})$ — Mansour [15].

- Permutations with at most r copies of the blocked permutation 13-2 — Claesson and Mansour [9][7].

3   Pin Sequences

Given points $p_1, \ldots, p_m$ in the plane, we denote by $\mathrm{rect}(p_1, \ldots, p_m)$ the smallest axes-parallel rectangle containing them.

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[5]*Blocked permutations*, introduced by Babson and Steingrímsson [4], are permutations containing dashes indicating the entries that need not occur consecutively. For example, 51342 contains two copies of 3-12: 513 and 534, but note that 514 is not a copy of 3-12 because the 1 and 4 are not adjacent.

[6]We have thus far ignored the other case; $\mathrm{Av}(123)$, and thus $\mathrm{Av}(123^{\leq r})$, contains infinitely many simple permutations, so these methods do not apply. The class $\mathrm{Av}(123)$ is enumerated by the Catalan numbers, $\mathrm{Av}(123^{\leq 1})$ was counted by Noonan [19], while $\mathrm{Av}(123^{\leq 2})$ was counted by Fulmek [13], proving a conjecture of Noonan and Zeilberger [20]. No results for larger values are known, although Fulmek conjectures formulas for $r = 3$ and $r = 4$ and further conjectures that $\mathrm{Av}(123^{\leq r})$ has an algebraic generating function for all r.

[7]While avoiding 132 and avoiding 13-2 are equivalent conditions, a permutation will tend to have fewer copies of 13-2.
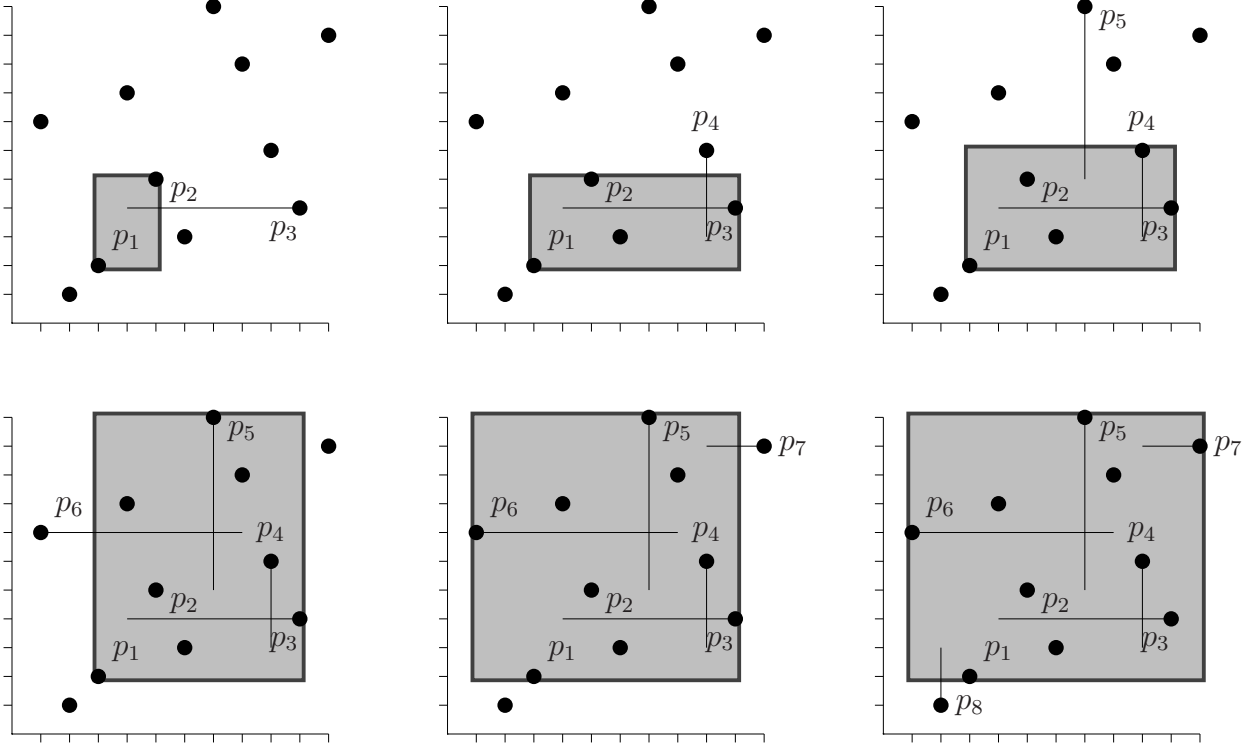
Figure 2: A pin sequence.

Take $\pi \in S_n$ and choose two points p_1 and p_2 in the plot of π. If these two points do not form an interval then there is at least one point which lies outside $\mathrm{rect}(p_1, p_2)$ and slices $\mathrm{rect}(p_1, p_2)$ either horizontally or vertically. (This discussion is accompanied by the sequence of diagrams shown in Figure 2.) We call such a point a *pin*. Choose a pin and label it p_3. Now consider the larger rectangle $\mathrm{rect}(p_1, p_2, p_3)$. If this also does not form an interval in π then we can find another pin, p_4, which slices $\mathrm{rect}(p_1, p_2, p_3)$ either horizontally or vertically. Again, if $\mathrm{rect}(p_1, p_2, p_3, p_4)$ is not an interval then we can find another pin p_5. We refer to a sequence of pins constructed in this manner as a *pin sequence*.

Formally, a *pin sequence* is a sequence of points p_1, p_2, ... in the plot of π such that for each $i \geq 3$,

- $p_i \notin \mathrm{rect}(p_1, \ldots, p_{i-1})$, and

- if $\mathrm{rect}(p_1, \ldots, p_{i-1}) = [a, b] \times [c, d]$ and $p_i = (x, y)$, we have either $a < x < b$ or $c < y < d$, or, in other words, p_i *slices* $\mathrm{rect}(p_1, \ldots, p_{i-1})$ either horizontally or vertically.

We describe pins as either *left*, *right*, *up*, or *down* based on their position relative to the rectangle that they slice. Thus in the pin sequence from Figure 2, p_3 and p_7 are right pins, p_4 and p_5 are up pins, p_6 is a left pin, and p_8 is a down pin (p_1 and p_2 lack direction).

A *proper pin sequence* is one that satisfies two additional conditions:

- *Maximality condition*: each pin must be maximal in its direction. For example, if $\mathrm{rect}(p_1, \ldots, p_{i-1}) = [a, b] \times [c, d]$ and $p_i = (x, y)$ is a right pin, then it is the right-most
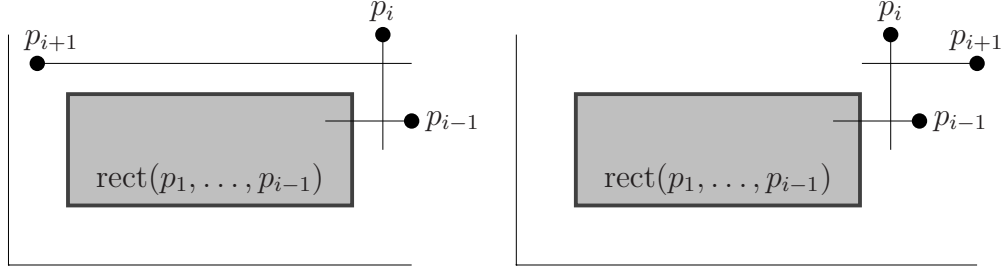
Figure 3: The two cases in the proof of Lemma 3.3.

of all possible right pins for this rectangle, or, in other words, the region $(x, n] \times [c, d]$ is devoid of points.

- *Separation condition*: p_{i+1} must *separate* p_i from $\{p_1, \ldots, p_{i-1}\}$. That is, p_{i+1} must lie horizontally or vertically between $\mathrm{rect}(p_1, \ldots, p_{i-1})$ and p_i.

For example, in the pin sequence shown in Figure 2, the choice of p_4 violates the maximality condition, while the choices of p_5, p_7, and p_8 violate the separation condition. The ultimate goal of the following succession of lemmas is to show (in Theorem 3.4) that all or all but one of the pins in a proper pin sequence themselves form a simple permutation. We begin by observing that proper pin sequences travel by $90°$ turns only.

Lemma 3.1. *In a proper pin sequence, p_{i+1} cannot lie in the same or opposite direction as p_i (for all $i \geq 3$).*

Proof. By the maximality condition, p_{i+1} cannot lie in the same direction as p_i. It cannot lie in the opposite direction by the separation condition. □

Lemma 3.2. *In a proper pin sequence, p_i does not separate any two members of $\{p_1, \ldots, p_{i-2}\}$.*

Proof. If p_i did separate $\mathrm{rect}(p_1, \ldots, p_{i-2})$ into two parts then p_{i-1} would lie on one side of this divide, violating the separation condition. □

Lemma 3.3. *In a proper pin sequence, p_i and p_{i+1} are separated either by p_{i-1} or by each of $p_1, \ldots, p_{i-2}$.*

Proof. The lemma is vacuously true for $i = 1$ and $i = 2$, so let us assume that $i \geq 3$. Without loss we may assume that p_{i-1} is a right pin and p_i is an up pin. By Lemma 3.1, p_{i+1} must be either a right pin or a left pin. The remainder of the proof is evident from Figure 3. □

We are now ready to prove our main result about proper pin sequences.

Theorem 3.4. *If $p_1, \ldots, p_m$ is a proper pin sequence then one of the sets of points $\{p_1, \ldots, p_m\}$, $\{p_1, \ldots, p_m\} \setminus \{p_1\}$, or $\{p_1, \ldots, p_m\} \setminus \{p_2\}$ is order isomorphic to a simple permutation.*

Proof. Suppose $m \geq 4$, as the smaller cases are trivially true. We are interested in the possible intervals in the subsequence given by the pins $p_1, \ldots, p_m$; we shall call these *intervals of pins*. Take $M \subseteq \{p_1, \ldots, p_m\}$ to be a minimal non-singleton interval of pins. Note that M is therefore order isomorphic to a simple permutation.

If M contains a pair of pins p_i and p_j with $i < j < m$ then by the separation condition $p_{j+1}, \ldots, p_m \in M$. Furthermore, because $j < m$, Lemma 3.3 shows that M contains either p_{j-1} or $p_1, p_2, \ldots, p_{j-2}$. In the latter case, if $j \geq 4$ then separation gives $p_{j-1} \in M$, as desired, while if $j \leq 3$, we have already found a minimal non-singleton interval of pins of the desired form. In the former case, the proof is completed by iterating this process.

Only the case $M = \{p_i, p_m\}$ remains. If $m - 1$, then Lemma 3.3 gives a contradiction. If $3 \leq i \leq m - 2$ then, by the separation condition, p_i separates $\{p_1, \ldots, p_{i-1}\}$, while Lemma 3.3 shows that p_m does not separate these points; thus at least one of them must lie in M, another contradiction.

We are now reduced to the cases $M = \{p_1, p_m\}$ and $M = \{p_2, p_m\}$. We consider the former; the latter is analogous. Because p_3 separates p_2 from p_1, it also separates p_2 from p_m, so $\{p_2, p_m\}$ cannot be an interval. If there are any other minimal non-singleton intervals of pins, then we are done by the considerations above. Therefore, $\{p_1, p_m\}$ is the only minimal non-singleton interval of pins, and thus $\{p_2, \ldots, p_m\}$ is order isomorphic to a simple permutation. □

As a corollary of this theorem, we see that Theorem 1.1 (in fact, a stronger result) is true for simple permutations with long pin sequences.

Corollary 3.5. *If π contains a proper pin sequence of length at least $2k + 2$ then π contains two disjoint simple subsequences, each of length at least k.*

Proof. Apply Theorem 3.4 to the two pin sequences $p_1, \ldots, p_{k+1}$ and $p_{k+2}, \ldots, p_{2k+2}$. □

We say that the pin sequence $p_1, \ldots, p_m$ for the permutation $\pi \in S_n$ is *saturated* if $\mathrm{rect}(p_1, \ldots, p_m) = [n] \times [n]$. For example, the pin sequence in Figure 2 is saturated. Any two points $p_1 \neq p_2$ in the plot of a simple permutation can be extended to a saturated pin sequence, as we are forced to stop extending a pin sequence only upon finding an interval or when the rectangle contains every point in π.

It is important to note that two points in a simple permutations need not be extendable to a proper saturated pin sequence. For example, the permutation in Figure 2 does not have a proper saturated pin sequence beginning with p_1 and p_2. For this reason we work with a weaker requirement: the pin sequence $p_1, \ldots, p_m$ is said to be *right-reaching* if p_m is the right-most point of π.

Lemma 3.6. *For every simple permutation π and pair of points p_1 and p_2 (unless, trivially, p_1 is the right-most point of π), there is a proper right-reaching pin sequence beginning with p_1 and p_2.*

Proof. Clearly we can find a saturated pin sequence $p_1, p_2, \ldots$ in π that satisfies the maximality condition. Since this pin sequence is saturated, it includes the right-most point; label it p_{i_1}. Now take i_2 as small as possible so that $p_1, p_2, \ldots, p_{i_2}, p_{i_1}$ is a valid pin sequence. Note first that $i_2 < i_1$ because $p_1, \ldots, p_{i_1}$ is a valid pin sequence. Now observe that p_{i_1} separates p_{i_2} from $\mathrm{rect}(p_1, \ldots, p_{i_2-1})$, because $p_1, \ldots, p_{i_2-1}, p_{i_1}$ is not a valid pin sequence.
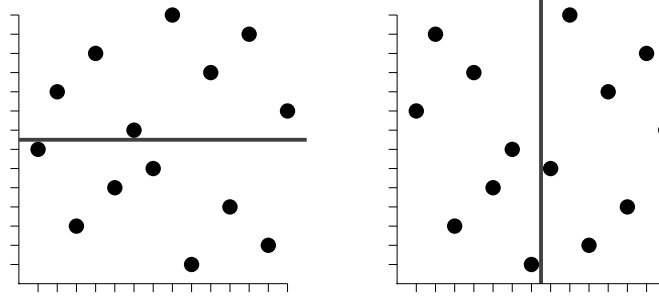
Figure 4: A vertical alternation (left) and its inverse, a horizontal alternation (right).
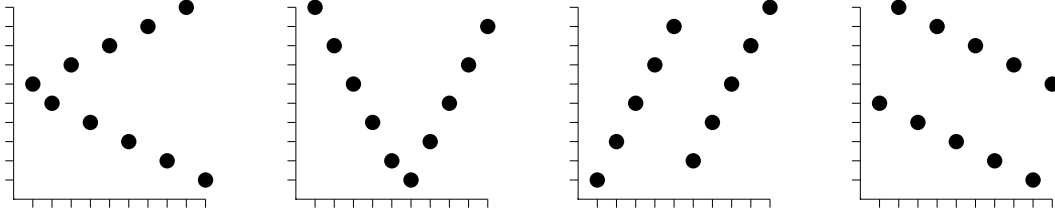

Figure 5: The two permutations on the left are wedge alternations, the two on the right are parallel alternations.

Continuing in this manner, we find pins p_{i_3}, p_{i_4}, and so on, until we reach the stage where $p_{i_{m+1}} = p_2$. Then $p_1, p_2, p_{i_m}, p_{i_{m-1}}, \ldots, p_{i_1}$ is a proper right-reaching pin sequence. $\qquad\square$

4   Simple permutations without long proper pin sequences

It remains only to consider simple permutations without long proper pin sequences, a consideration which constitutes the bulk of the proof. Our goal in this section is to prove that these permutations contain long "alternations". A *horizontal alternation* is a permutation in which every odd entry lies to the left of every even entry, or the reverse of such a permutation. A *vertical alternation* is the group-theoretic inverse of a horizontal alternation. Examples are shown in Figure 4.

Every sufficiently long vertical alternation contains either a long *parallel alternation* or a long *wedge alternation* (see Figure 5 for definitions):

Proposition 4.1. *Every alternation of length at least $2k^4$ contains either a parallel or wedge alternation of length at least $2k$.*

Proof. Let π be a vertical alternation of length $2n \geq 2k^4$. By the Erdős-Szekeres Theorem (every permutation of length n contains a monotone subsequence of length at least $\sqrt{n}$), the sequence $\pi(1), \pi(3), \ldots, \pi(2n-1)$ contains a monotone subsequence of length at least k^2, say $\pi(i_1), \pi(i_2), \ldots, \pi(i_{k^2})$. Applying the Erdős-Szekeres Theorem to the subsequence $\pi(i_1 + 1), \pi(i_2 + 1), \ldots, \pi(i_{k^2} + 1)$ completes the proof. $\qquad\square$
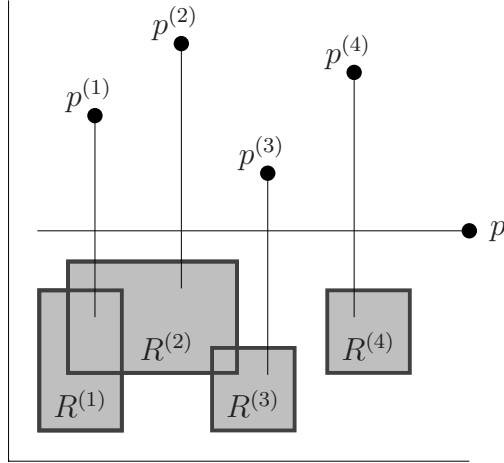
Figure 6: The situation that arises in the proof of Lemma 4.2.

Note that every parallel alternation of length $2k + 2 \geq 10$ contains two disjoint simple permutations of length at least k. Thus Theorem 1.1 follows in the case where our simple permutation contains a long parallel alternation.

We say that the pin sequences $p_1, \ldots, p_s$ and $q_1, \ldots, q_t$ *converge at the pin* x if there exist $i, j \geq 3$ so that $p_i = q_j = x$ but $p_{i-1} \neq q_{j-1}$.

Lemma 4.2. *If $8k$ proper pin sequences of π converge at the same pin, then π contains an alternation of length at least $2k$.*

Proof. Let us suppose that $8k$ pin sequences converge at the pin p. This pin could be variously functioning as a left, right, down, or up pin for each of these $8k$ sequences, but p plays the same role for at least $2k$ sequences. Suppose, without loss, that p is a right pin for at least $2k$ sequences. Now consider the immediate predecessors to p in these sequences. These pins can be either up pins or down pins (by Lemma 3.1). By symmetry, we may assume that for at least k of these pin sequences the immediate predecessor to p is an up pin. Reading left to right, label these immediate predecessor pins $p^{(1)}, p^{(2)}, \ldots, p^{(k)}$ and let $R^{(i)}$ denote the rectangle for which $p^{(i)}$ is a pin. Note that each $R^{(i)}$ lies completely below p, as otherwise the separation condition would prevent p from following $p^{(i)}$ in the corresponding pin sequence. We now have the situation depicted in Figure 6.

It suffices to show, for each i, that π contains a point lying horizontally between $p^{(i)}$ and $p^{(i+1)}$ and below p, since then these points, together with the $p^{(i)}$'s and p, will give an alternation of length $2k$. However, if there is no such point then $p^{(i)}$ and $p^{(i+1)}$ could each function as up pins for both $R^{(i)}$ and $R^{(i+1)}$, and thus one of these choices would contradict the maximality condition, completing the proof. ∎

Lemma 4.3. *Every simple permutation of length at least $2(8k^4)^{(8k^4)^{2k}}$ contains either a proper pin sequence of length at least $2k$ or a parallel or wedge alternation of length at least $2k$.*

Proof. Suppose that the simple permutation $\pi \in S_n$ contains neither a proper pin sequence of length at least $2k$ nor a parallel or wedge alternation of length at least $2k$. In particular,
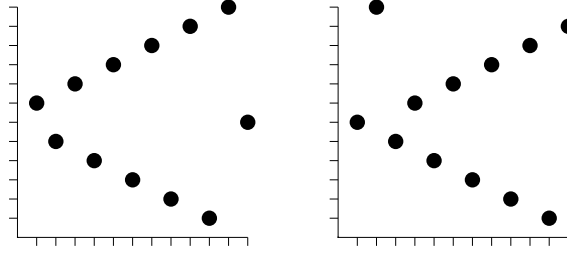
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Figure 7: The two types of wedge simple permutations. On the left, type 1, on the right, type 2.

π does not contain a proper right-reaching pin sequence of length $2k$, and it follows from Proposition 4.1 that π has no alternations of length $2k^4$.

Pair up each of the entries of π except the right-most. Taking proper right-reaching pin sequences beginning at each of these pairs creates $\lfloor (n-1)/2 \rfloor$ sequences.

As these pin sequences are right-reaching, they all agree on their final (right-most) pin which we denote by p. By Lemma 4.2, fewer than $8k^4$ of these pin sequences converge at p; equivalently, there are fewer than $8k^4$ immediate predecessors to p. Label these immediate predecessors $p^{(1)}, p^{(2)}, \ldots, p^{(m)}$. Again, fewer than $8k^4$ pin sequences converge at each of the $p^{(i)}$'s, so there are fewer than $(8k^4)^{8k^4}$ immediate predecessors to these pins. Continue this process until we reach the sequences of length $2k$, of which we have assumed there are none. We have thus counted all $\lfloor (n-1)/2 \rfloor$ of our sequences, and have obtained the bound

$$\left\lfloor \frac{n-1}{2} \right\rfloor < 1 + 8k^4 + (8k^4)^{8k^4} + (8k^4)^{(8k^4)^2} + \cdots + (8k^4)^{(8k^4)^{(2k-1)}},$$

so, simplifying,

$$n < 2(8k^4)^{(8k^4)^{2k}}. \qquad \square$$

We are left to deal with simple permutations which do not have long proper pin sequences but do have long wedge alternations. We prove that these permutations contain long *wedge simple permutations*, of which there are two types (up to symmetry). Examples of these two types are shown in Figure 7.

Lemma 4.4. *If a simple permutation contains a wedge alternation of length $4k^2$ then it contains either a pin sequence of length at least $2k$ or a wedge simple permutation of length at least $2k$.*

Proof. Let π be a simple permutation containing a wedge alternation of length at least $4k^2$. By symmetry we may assume that this wedge alternation opens to the right (i.e., it is oriented as $<$). We call these the *wedge points* of π. Label the two left-most wedge points p_1 and p_2 and by Lemma 3.6 extend this into a proper right-reaching pin sequence $p_1, p_2, \ldots, p_m$.

Let R_i denote the smallest rectangle in the plot of π containing p_1, p_2, and p_i that is not sliced by a wedge point outside the rectangle. Define the *wedge sum* of the pin p_i, $\mathrm{ws}(p_i)$, to be the number of wedge points in R_i. For $i \geq 2$ define the *wedge contribution* of p_i by $\mathrm{wc}(p_i) = \mathrm{ws}(p_i) - \mathrm{ws}(p_{i-1})$ and set $\mathrm{wc}(p_1) = 1$. Regarding these quantities we make four observations:
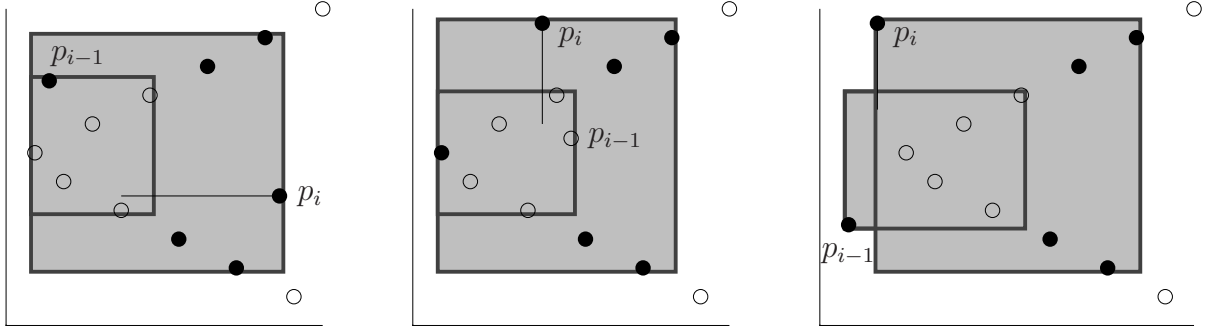
Figure 8: The three cases in the proof of Lemma 4.4; the solid points form simple permutations.

(W1) the wedge sum of p_m is equal to the total number of wedge points and also to
$$\sum_{i=1}^{m} \mathrm{wc}(p_i),$$

(W2) it is not hard to construct examples in which pins have negative wedge contributions; indeed,

(W3) left pins cannot have positive wedge contributions, and finally,

(W4) if p_i is an up pin, then the right-most wedge point in R_i is an upper wedge point.

We now claim that each p_i lies in a wedge simple permutation of length at least $\mathrm{wc}(p_i) + 2$. This claim implies the theorem, because if no pin lies in a wedge simple permutation of length at least $2k$ then $\mathrm{wc}(p_i) \le 2k - 3$, so by (W1),

$$4k^2 \le \sum_{i=1}^{m} \mathrm{wc}(p_i) \le m(2k - 3),$$

and thus $m \ge 2k$, giving the long pin sequence desired.

The claim is easily observed for $i = 1$ and, by (W3), vacuously true if p_i is a left pin. Thus by symmetry there are only three cases to consider: an up pin followed by a right pin, a right pin followed by an up pin, and a left pin followed by an up pin. These three cases are depicted in Figure 8.

Let us consider in detail the case of an up pin followed by a right pin. By (W4), the left-most wedge point in $R_i \setminus R_{i-1}$ lies below p_1. By separation, p_{i-1} lies above p_i, which is itself the right-most point in R_i. Therefore the wedge points in $R_i \setminus R_{i-1}$ together with p_i and p_{i-1} constitute a type 1 wedge simple permutation. The other cases follow by similar analysis; in the right-up case the wedge points in $R_i \setminus R_{i-1}$ together with p_1 and p_i give a wedge simple permutation of type 2, while in the left-up case a wedge simple permutation of type 2 can be formed from the wedge points in $R_i \setminus R_{i-1}$, p_{i-1}, and p_i. $\square$

We have therefore established the following theorem.

Theorem 4.5. *Every simple permutation of length at least $2(2048k^8)^{(2048k^8)^{2k}}$ contains a proper pin sequence of length $2k$, a parallel alternation of length $2k$, or a wedge simple permutation of length $2k$.*

The proof of Theorem 1.1 now follows by analysing each of these cases in turn. A parallel alternation of length $2k + 2 \geq 10$ contains two disjoint simple permutations of length k. A type 1 wedge simple permutation of length $2k$ contains two type 1 wedge simple permutations of length k with only one entry in common, and a type 2 wedge simple permutation of length $2k$ contains two type 2 wedge simple permutations of length k which share two entries. Finally, Corollary 3.5 shows that a permutation with a proper pin sequence of length $2k + 2$ contains two disjoint simple permutations of length k.

Brignall, Ruškuc, and Vatter [8] apply Theorem 4.5 to show that it is possible to decide whether or not a permutation class contains only finitely many simple permutations, and expatiate upon Lemma 4.3, showing that every long simple permutation contains either a long alternation or a long "oscillation".

5   Other contexts

Although our proof is highly permutation-centric, these is no reason why analogues of Theorem 1.1 cannot exist for other types of object. For example, an interval[8] in a graph is a set of vertices $X \subseteq V(G)$ such that $N(v) \setminus X = N(w) \setminus X$ for all $v, w \in X$, where $N(v)$ denotes the neighbourhood of v in G. A graph on n vertices therefore has several trivial intervals ($\emptyset$, $V(G)$, and the singletons); a graph with no nontrivial intervals is then often called *prime* or *indecomposable* (the word simple meaning something completely different in this context). These graphs have been the subject of considerable study, see Ehrenfeucht, Harju, and Rozenberg [11], Ille [14], and Sabidussi [21].

The most general context for simplicity — and thus the most general context for results such as Theorem 1.1 — is relational structures. Let $\mathcal{L}$ denote a relational language (i.e., a set of relational symbols together with positive integers n_R for each relational symbol $R \in \mathcal{L}$, specifying the arity of R) and $\mathcal{A}$ an $\mathcal{L}$-structure (i.e., a ground set $\mathrm{dom}(\mathcal{A})$ together with interpretations of the relational symbols from $\mathcal{L}$). Following Földes [12], we say that the subset $X \subseteq \mathrm{dom}(\mathcal{A})$ is an interval if the following occurs for every relation $R \in \mathcal{L}$ and every n_R-tuple $(x_1, x_2, \ldots, x_{n_R}) \in \mathrm{dom}(\mathcal{A})^{n_R} \setminus X^{n_R}$: if $x_i \in X$ then the value of $R^{\mathcal{A}}(x_1, x_2, \ldots, x_{n_R})$ is unchanged by swapping x_i with any other element of X. Again the relational structure $\mathcal{A}$ will have the trivial intervals $\emptyset$, $\{a\}$ for all $a \in \mathrm{dom}(\mathcal{A})$, and $\mathrm{dom}(\mathcal{A})$ itself, and it is simple if it has no others.

In this most general context, any analogue of Theorem 1.1 would need to allow for more intersection between the two simple substructures. An example demonstrating this is given in Footnote 9.

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[8]These are also called strong intervals, partie solidaires, blocks, factors, modules, clans, congruences, and convex subsets.

[9]Let $\mathcal{L}$ consist of a 2-ary relation $<$ and a k-ary relation R. Take $\mathcal{A}$ with $\mathrm{dom}(\mathcal{A}) = [2n]$ where $<$ is interpreted as the normal linear order on $[2n]$ and $R(1, 3, 5, \ldots, 2k-3, i)$ precisely for even $i \in [2k-2, 2n]$. This structure is simple, but all simple substructures (with at least two elements) of $\mathcal{A}$ must contain each

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of $1, 3, 5, \ldots, 2k - 3$, and then to prevent these elements from containing a nontrivial interval, the simple substructure must also contain $2, 4, 6, \ldots, 2k - 4$.

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